BOUNDARY VALUE PROBLEM OF THE THEORY OF CREEP FOR A BODY WITH ACCRETION

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Papers [1, 2] are used as the basis for the formulation of a boundary value problem of the theory of creep for a nonuniformly aging body, with the elements of varying ages accumulating on the body in a continuous or discrete manner. The initial equations are given and conditions formulated which determine the solution of the boundary value problem of the theory of creep for such bodies. The characteristic feature of these bodies is, that in the course of accumulation not only their form, surface and volume forces and the boundary conditions change, but also their physical and mechanical properties with respect to time and the coordimates.

This is due to the fact that the aging process in these bodies does not follow the same course in all their elements. Such phenomena take place during a consecutive erection and loading of engineering structures, in the crystal growing processes, during the phase transitions in viscoelastic bodies, etc.

The basic papers dealing with solutions of the problems of accretion using the methods of the theory of elasticity are given in [3]. The theory of elastic body with creep is used in [4] to study the stress-strain state in homogeneous bodies with accretion. A more general formulation is used for the same problem in [5].

1. Formulation of the problem and the derivation of the solution equations for the case of discrete accumulation on non-

uniformly aging bodies. Let k isotropic bodies be given occupying the regions $\Omega^{(i)}$ (i = 1, 2, ..., k). The material of these bodies has the property of aging as well as of creep. It is known that the bodies are produced at the instants of time τ_i^* and loaded at the instants of time τ_i° (i = 1, 2, ..., k). We further assume that at a certain instant t_{ij} the body $\Omega^{(i)}$ is joined to the body $\Omega^{(j)}$ along a certain surfaces S_{ij} . Since some of the bodies may remain disjoined, t_{ij} can be specified for only certain values of i and j. We shall arrange the set of values of t_{ij} in increasing order, denoting the terms of the new sequence by t_m $(m = 1, 2, 3, \ldots, M)$. It is not assumed here that all regions $\Omega^{(i)}$ have been specified prior to the first instance of joining these bodies. It is only necessary to assume that $t_{ij} > \tau_i^*$ and $t_{ij} > \tau_j^*$.

Let us denote by $\Omega(t)$ the region occupied by the union of the regions $\Omega^{(i)}$ of bodies produced up to the instant t, i.e.

$$\Omega(t) = \bigcup_{i} \Omega^{(i)}$$
(1.1)

for *i* such that $\tau_i^* \leqslant t$.

Thus the region $\Omega(t)$ changes at the instants of time $t = \tau_i^*$, i.e. when the next body in turn is produced, and at the instant of time $t = t_{ij} = t_m$, i.e. when the bodies merge. The surface of contact S_{ij} is assumed to be stress-free up to the instant of merger

$$\sigma_n (\mathbf{r}, t) = 0, \quad \mathbf{r} \in S_{ij} \tag{1.2}$$

where $\sigma_n(\mathbf{r}, t)$ is the stress vector at the surface element with outer normal \mathbf{n} .

We denote the components of the displacement vectors, deformation and stress tensors in the body $\Omega^{(i)}$ up to the instant of merger and after the merger, by $u_{\alpha}{}^{(i)}$, $\varepsilon_{\alpha\beta}{}^{(i)}$, $\sigma_{\alpha\beta}{}^{(i)}$, $\sigma_{\alpha\beta}{}^{(i)$

 $\Omega^{(i)}$ and loaded at the instants of time τ_i° can be written, in accordance with [1,2], in the form

$$\begin{aligned} \varepsilon_{\alpha\beta}^{(i)}(\mathbf{r},t) &= \varepsilon_{\alpha\beta}^{\circ(i)}(\mathbf{r},t) + (1+\nu) \left[(I+L^{(i)}) \frac{\sigma_{\alpha\beta}^{(i)}}{E} \right] - \\ \nu \delta_{\alpha\beta} \left[(I+L^{(i)}) \frac{\sigma_{ss}^{(i)}}{E} \right] \\ \sigma_{\alpha\beta}^{(i)}(\mathbf{r},t) &= \frac{E(t-\tau_i^*)}{1+\nu} \left\{ (I+N^{(i)}) \left[\varepsilon_{\alpha\beta}^{(i)} - \varepsilon_{\alpha\beta}^{\circ(i)} \right] + \\ \delta_{ij} \frac{\nu}{1-2\nu} (I+N^{(i)}) \left[\varepsilon_{ss}^{(i)} - \varepsilon_{ss}^{\circ(i)} \right] \right\} \\ I \left(\frac{\sigma_{\alpha\beta}^{(i)}}{E} \right) &= \frac{\sigma_{\alpha\beta}^{(i)}(\mathbf{r},t)}{E(t-\tau_i^*)} \\ L^{(i)} \left(\frac{\sigma_{\alpha\beta}^{(i)}}{E} \right) &= \int_{\tau_i^{\circ}}^{t} \frac{\sigma_{\alpha\beta}^{(i)}(\mathbf{r},\tau)}{E(\tau-\tau_i^*)} P(t-\tau_i^*,\tau-\tau_i^*) d\tau \\ N^{(i)} \left(\varepsilon_{\alpha\beta}^{(i)} \right) &= \int_{\tau_i^{\circ}}^{t} \varepsilon_{\alpha\beta}^{(i)}(\mathbf{r},\tau) R(t-\tau_i^*,\tau-\tau_i^*) d\tau \end{aligned}$$

Here v is the Poisson's ratio $\delta_{\alpha\beta}$ is the Kronecker delta, $\varepsilon_{\alpha\beta}^{\circ(i)}$ denote the forced deformations and $E(t - \tau_i^*)$, $P(t - \tau_i^*, \tau - \tau_i^*)$, $R(t - \tau_i^*, \tau - \tau_i^*)$ are, respectively, the modulus of instantaneous elastic deformation, the creep kernel and the relaxation kernel of the material of the body occupying the region $\Omega^{(i)}$ and produced at the instant of time τ_i^* .

The fundamental condition concerning the merger of the body is, that no disconti-

nuities will appear in the displacements arising in the body during the interval of time between two consecutive mergers, and in the stress vector σ_n at the contact surface.

The requirement of continuity of the displacement ensures that no cracks appear between the merging bodies and no slippage takes place along the boundary surface. This can be expressed in mathematical terms as follows. The displacements u_{α} (r, t) of the points of the body occupying the region Ω (t) represent, at $t_m \leq t < t_{m+1}$ the sum of the displacements u_{α} (r, t_m) at the instant $t = t_m$ (in general, these displacements will have undergone jumps at the surface of contact at the instants of time t_1, t_2, \ldots, t_m) and of the displacement increments $\Delta^{(m)} u_{\alpha}$ (r, t) satisfying the condition of continuity in the region Ω (t) for $t_m \leq t < t_{m+1}$

$$u_{\alpha}(\mathbf{r},t) = u_{\alpha}(\mathbf{r},t_m) + \Delta^{(m)} u_{\alpha}(\mathbf{r},t)$$
(1.4)

Let us write a system of equation defining a solution of the boundary value problem for the region $\Omega(t)$, i.e. for the region embracing *m* nonuniformly aging bodies undergoing consecutive incremental growth. The equation connecting the deformations and the displacements after the merger, namely when $t_m \ll t < t_{m+1}$ is, in accordance with (1.4),

$$\varepsilon_{\alpha\beta}(\mathbf{r},t) = \varepsilon_{\alpha\beta}(\mathbf{r},t_m) + \frac{1}{2} \left[\Delta^{(m)} u_{\alpha,\beta}(\mathbf{r},t) + \Delta^{(m)} u_{\beta,\alpha}(\mathbf{r},t) \right]$$
(1.5)

where $\Delta^{(m)}u_{\alpha}$ denote the displacement increments satisfying the conditions of continuity in the region $\Omega(t)$ for $t_m \ll t < t_{m+1}$ and determined together with the remaining unknown functions of the problem in question.

Let us denote the boundary of $\Omega(t)$ by S(t), and let this boundary consist of five segments

$$S(t) = S_0(t) \cup S_1(t) \cup S_2(t) \cup S_3(t) \cup S_4(t)$$

The segment $S_0(t)$ is stress-free and represents the surface on which the consecutive accretion will take place. The stresses $\mathbf{F}(\mathbf{r}, t) = \{F_{\alpha}(\mathbf{r}, t)\}$ are specified on $S_1(t)$, the displacements $\mathbf{V}(\mathbf{r}, t) = \{V_{\alpha}(\mathbf{r}, t)\}$ on $S_2(t)$, the normal displacements $V_n(\mathbf{r}, t)$ and the tangential stress vector $\mathbf{F}_{\tau}(\mathbf{r}, t)$ on $S_3(t)$, the stresses $F_n(\mathbf{r}, t)$ and the tangential displacement vector $\mathbf{V}_{\tau}(\mathbf{r}, t)$ on $S_4(t)$. In the region $\Omega(t)$ we have the volume forces $f(\mathbf{r}, t) = \{f_{\alpha}(\mathbf{r}, t)\}$ and the forced deformations $\varepsilon_{\alpha\beta}^{\circ}(\mathbf{r}, t)$. Then the equations of quasi-static equilibrium and the boundary conditions for the stresses and displacements written for the instant $t_m \leqslant t < t_{m+1}$, will have the form

$$\sigma_{\alpha\beta,\beta}(\mathbf{r},t) + f_{\alpha}(\mathbf{r},t) = 0, \quad \mathbf{r} \in \Omega(t)$$

$$\sigma_{n}(\mathbf{r},t) = 0, \quad \mathbf{r} \in S_{0}(t)$$

$$\sigma_{n}(\mathbf{r},t) = \mathbf{F}(\mathbf{r},t), \quad \mathbf{r} \in S_{1}(t)$$

$$\mathbf{u}(\mathbf{r},t) = \mathbf{V}(\mathbf{r},t), \quad \mathbf{r} \in S_{2}(t)$$

$$u_{n}(\mathbf{r},t) = V_{n}(\mathbf{r},t), \quad \sigma_{\tau}(\mathbf{r},t) = \mathbf{F}_{\tau}(\mathbf{r},t), \quad \mathbf{r} \in S_{3}(t)$$

$$\sigma_{n}(\mathbf{r},t) = F_{n}(\mathbf{r},t), \quad \mathbf{u}_{\tau}(\mathbf{r},t) = \mathbf{V}_{\tau}(\mathbf{r},t), \quad \mathbf{r} \in S_{4}(t)$$
(1.6)

Let us introduce the function $\gamma(\mathbf{r}) = \tau_i^{\circ}$ for $\mathbf{r} \in \Omega^{(i)}$ and $\varkappa(\mathbf{r}) = \tau_i^{*}$ for

 $\mathbf{r} \in \Omega^{(i)}$. Then the equation of state of the theory of creep connecting the tensors $e_{\alpha\beta}(\mathbf{r}, t)$ and $\sigma_{\alpha\beta}(\mathbf{r}, t)$ will have the form

$$\begin{aligned} \varepsilon_{\alpha\beta}(\mathbf{r},t) &= (1+\nu)\left[I+L\right]\left(\frac{\sigma_{\alpha\beta}}{E}\right) - \nu\delta_{\alpha\beta}\left[I+L\right]\left(\frac{\sigma_{ss}}{E}\right) + \varepsilon_{\alpha\beta}^{\bullet}(\mathbf{r},t) \quad (1.8) \\ \sigma_{\alpha\beta}(\mathbf{r},t) &= \frac{E\left[t-\varkappa(\mathbf{r})\right]}{1+\nu} \left\{ (I+N)\left(\varepsilon_{\alpha\beta} - \varepsilon_{\alpha\beta}^{\bullet}\right) + \delta_{\alpha\beta}\frac{\nu}{1-2\nu} \left(I+N\right)\left(\varepsilon_{ss} - \varepsilon_{ss}^{\circ}\right) \right\} \\ I\left(\frac{\sigma}{E}\right) &= \frac{\sigma_{\alpha\beta}(\mathbf{r},t)}{E\left[t-\varkappa(\mathbf{r})\right]} \\ L\left(\frac{\sigma_{\alpha\beta}}{E}\right) &= \int_{\gamma(\mathbf{r})}^{t} \frac{\sigma_{\alpha\beta}(\mathbf{r},\tau)}{E\left[\tau-\varkappa(\mathbf{r})\right]} P\left[t-\varkappa(\mathbf{r}), \tau-\varkappa(\mathbf{r})\right] d\tau \\ N\left(\varphi\right) &= \int_{\gamma(\mathbf{r})}^{t} R\left[t-\varkappa(\mathbf{r}), \tau-\varkappa(\mathbf{r})\right] \varphi\left(\mathbf{r},\tau\right) d\tau \end{aligned}$$

We note that $R [t - \varkappa (\mathbf{r}), \tau - \varkappa (\mathbf{r})]$ is a resolvent of the creep kernel $P[t - \varkappa (\mathbf{r}), \tau - \varkappa (\mathbf{r})]$ and represents the relaxation kernel of the material of the body.

The system of equation consisting of (1, 5), the first equation of (1, 6) and the first equation of (1, 8) with the boundary conditions (1, 7), represents a closed system which determines the solution of the boundary value problem of the theory of creep for nonuniformly aging bodies undergoing incremental growth.

It should be noted that when the body forces f_{α} are bounded in the neighborhood of the surface of merger S_{ij} with the normal \mathbf{n}_{σ} the equations of equilibrium (1.6) yield the conditions of continuity on S_{ij} of the normal $\sigma_n = \sigma_{\alpha\beta}n_{\alpha}n_{\beta}$ and tangential $\sigma_{\tau} = \sigma_n - \sigma_n \mathbf{n}$ components of the stress tensor. However, since the remaining components of the stress tensor $\sigma_{\alpha\beta}$ may, generally speaking, suffer discontinuities at the merger surfaces, the derivatives in the equations of equilibrium (1.6) must be interpreted in the generalized, instead of the usual sense.

Moreover, since the boundary value problem under consideration is geometrically linear and the conditions of continuity of the displacement increments $\Delta^{(\hat{m})}u_x(\mathbf{r}, t)$ hold on the surface of merger S_{ij} in accordance with (1, 4) and (1, 5), therefore the solution obtained must satisfy, at each point \mathbf{r} of the surface S_{ij} , the conditions of continuity of the displacement component normal to the surface S_{ij} , and of the deformation tensor components defining the elongation and shear in the plane tangent to this surface at any point \mathbf{r} .

Thus the solution of the boundary value problem formulated here will automatically satisfy the usual conditions of continuity of the displacements, deformations and stresses at the boundary surface between the component bodies [6]. The jumps in the values of components of the stress field at the surface S_{ij} can be determined by solving the boundary value problem consisting of Eq. (1.5), the first equation of (1.6) and conditions (1.7). 2. Statement of the problem and initial equations of the theory of creep for nonuniformly aging bodies undergoing continuous accretion. Let a nonuniformly aging body the matrial of which exhibits the property of creep, occupy the region Ω^* . We know that the body is constructed at the instant of time t_0 and loaded at the instant $\tau_0 \ge t_0$. Further, from some instant of time $t^* > \tau_0$ onwards, the body undergoes a continuous accumulation of elements of varying ages.

We denote by $\Omega(t)$ the region occupied by the growing body at the time $t \ge t^*$ so that $\Omega(t^*) = \Omega$ and $\Omega(t_1) \subset \Omega(t_2)$ if $t_1 < t_2$. We denote the boundary of

 Ω^* by S^* and the boundary of $\Omega(t)$ by S(t). To simplify the notation, we define additionally $\Omega(t)$ and S(t) at $t \leq t^*$ as follows: $\Omega(t) \equiv \Omega^*$; $S(t) \equiv S^*$ when $t_0 \leq t \leq t^*$. We denote by τ_1^* (r) the instant of preparation of the accumulated element near the point $\mathbf{r} = \{x_1, x_2, x_3\}$ so that τ_1^* (r) = t_0 when

 $r \in \Omega^*.$ Then the equation of state for any point $% \Omega^*$ of the accumulated element will be

$$\varepsilon_{\alpha\beta}(\mathbf{r},t) = (1+\nu)(I+L)\left(\frac{\sigma_{\alpha\beta}}{E}\right) - \nu\delta_{\alpha\beta}(I+L)\left(\frac{\sigma_{ss}}{E}\right) + \dot{\varepsilon_{\alpha\beta}}(\mathbf{r},t) \quad (2.1)$$

where I is an identity operator defined in (1.8) and the operator L is given by the formula

$$L\left(\frac{\sigma_{\alpha\beta}}{E}\right) = \int_{\tau_0}^{t} \frac{\sigma_{\alpha\beta}\left(\mathbf{r},\tau\right)}{E\left[\tau - \tau_1^*\left(\mathbf{r}\right)\right]} P\left[t - \tau_1^*\left(\mathbf{r}\right), \tau - \tau_1^*\left(\mathbf{r}\right)\right] d\tau \qquad (2.2)$$

The relation connecting the deformation tensor $\varepsilon_{\alpha\beta}$ with the stress tensor $\sigma_{\alpha\beta}$ can be written in a different form. To do this we denote by $\tau_1(\mathbf{r})$ the age of the material of the element situated near the point $\mathbf{r} = \{x_1, x_2, x_3\}$ at the instant of observation t. Obviously

$$\tau_1(\mathbf{r}) = t - \tau_1^*(\mathbf{r}) \tag{2.3}$$

 $\tau_1^{\circ}(r) = \tau_0 - \tau_1^{*}(r)$ where $\tau_1^{\circ}(r)$ is the age of the material at the instant τ_0 of its loading.

Using the relation (2.3) and performing the change of variable $\xi = \tau - \tau_1^*$ (r) we can write the operator $L(\sigma_{\alpha\beta}/E)$ in the form

$$L\left(\frac{\sigma_{\alpha\beta}}{E}\right) = \int_{\tau_1^{\nu}(\mathbf{r})}^{\tau_1(\mathbf{r})} \frac{\sigma_{\alpha\beta}\left(\xi + \tau_1^*\left(\mathbf{r}\right)\right)}{E\left(\xi\right)} P\left[\tau_1\left(\mathbf{r}\right), \xi\right] d\xi$$
(2.4)

Thus the equation of state (2.1) includes, in the case of (2.4), the age of the material τ_1 (r).

Let us now turn our attention to the equation connecting the deformations and displacements in the body undergoing an accumulation. We derive these equations by approximating the process of continuous accumulation with help of the discrete accumulation and passing to the limit as the length of the longest time interval tends

to zero. Let Δt denote a short time interval. Setting $t_m = t^* + (m-1) \cdot \Delta t$ (m = 1, 2, ...) we introduce the incremental region $\Delta \Omega = \Omega (t_{m+1} / \Omega (t_m))$. Then

the process of discrete accumulation can be described in the following manner. When $t \in [t_1, t_2)$, the body occupies the initial region $\Omega^* = \Omega(t_1) = \Omega(t^*)$ and is deformed by the forces defined in the region $\Omega(t_1)$ and on its surface $S(t_1)$. At the instant t_2 the region $\Delta\Omega_1$ comes into existence and merges, at the same instant t_2 , with $\Omega(t_1)$ so that a new body forms occupying the region $\Omega(t_2) = \Omega(t_1) \cup \Delta\Omega_1$. Continuing this process we find that the body remains unchanged

over the interval $[t_m, t_{m+1})$ and occupies the region $\Omega(t_m)$. At the instant t_{m+1} a region $\Delta\Omega_m$ comes into existence and at once merges with the body occupying the region $\Omega(t_m)$ forming a new body occupying the region $\Omega(t_{m+1}) = \Omega(t_m) \cup \Delta\Omega_m$.

Returning now to the equations (1.4) and (1.5) connecting the deformations and displacements during the discrete accumulation, we note that the displacements and deformations in the region $\Omega(t_m)$ will be connected when $t \in [t_m, t_{m+1})$ by the following relations:

$$u_{\alpha}(\mathbf{r}, t) = u_{\alpha}(\mathbf{r}, t_{m}) + \Delta u_{\alpha}(\mathbf{r}, t), \quad \mathbf{r} \in \Omega (t_{m})$$

$$\varepsilon_{\alpha\beta}(\mathbf{r}, t) = \varepsilon_{\alpha\beta}(\mathbf{r}, t_{m}) + \frac{1}{2} [\Delta u_{\alpha, \beta}(\mathbf{r}, t) + \Delta u_{\beta, \alpha}(\mathbf{r}, t)]$$
(2.5)

where the displacement increment Δu_{α} (r, t) satisfies the conditions of continuity in the region $\Omega(t_m)$.

From (2.5) we obviously have

$$\lim_{\Delta t \to 0} \frac{u_{\alpha}(\mathbf{r}, t) - u_{\alpha}(\mathbf{r}, t_{m})}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta u_{\alpha}(\mathbf{r}, t)}{\Delta t} = u_{\alpha}(\mathbf{r}, t)$$

$$\dot{\varepsilon}_{\alpha\beta}(\mathbf{r}, t) = \lim_{\Delta t \to 0} \frac{\varepsilon_{\alpha\beta}(\mathbf{r}, t) - \varepsilon_{\alpha\beta}(\mathbf{r}, t_{m})}{\Delta t} = \frac{1}{2} \lim_{\Delta t \to 0} \left[\frac{\Delta u_{\alpha, \beta}(\mathbf{r}, t)}{\Delta t} + \frac{\Delta u_{\beta, \alpha}(\mathbf{r}, t)}{\Delta t} \right]$$

and the latter gives

$$\varepsilon_{\alpha\beta}(\mathbf{r},t) = \frac{1}{2} \left[u_{\alpha,\beta}(\mathbf{r},t) + u_{\beta,\alpha}(\mathbf{r},t) \right]$$
(2.6)

$$u_{\alpha} \cdot (\mathbf{r}, t) = \frac{\partial u_{\alpha}(\mathbf{r}, t)}{\partial t}, \quad \dot{e}_{\alpha\beta}(\mathbf{r}, t) = \frac{\partial e_{\alpha\beta}(\mathbf{r}, t)}{\partial t}$$
 (2.7)

Thus at any instant of time t the rates of deformation of a nonuniformly aging elastic body with creep undergoing a continuous accumulation, are connected with the displacement rates by the Cauchy relation (2.6).

The equations of quasi-static equilibrium of the body with accumulation occupying the region $\Omega(t)$ and the boundary conditions at its surface S(t), will have the usual form

$$\sigma_{\alpha\beta,\beta}(t) + f_{\alpha}(t) = 0, \quad \mathbf{r} \in \Omega(t)$$
(2.8)

$$\sigma_{\alpha\beta}n_{\beta} = F_{\alpha}(t), \quad \mathbf{r} \in S_{F}(t); \quad u_{\alpha} = V_{\alpha}(t), \quad \mathbf{r} \in S_{V}(t)$$
(2.9)

where $S_F(t)$ and $S_V(t)$ are the respective segments of the surface S(t) on which

the stresses and deformations are specified. Generally, they vary with time.

Equations (2, 1), the first equation of (2, 5), (2, 8) and the boundary conditions (2, 9), together form a complete system which defines the solution of the boundary value problem for a nonuniformly aging body with creep undergoing a continuous accumulation.

It is interesting to note that we can illustrate the theory of creep developed above for a body with accumulation, by considering a boundary value problem of the theory of viscoelasticity for the bodies the boundary surfaces of which vary with time as the result of the phase transformations [7].

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Translated by L.K.